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### **What if criminals optimize their choice? Optimal strategies to defend public goods**

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## Highlights

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► We look for optimal strategies to defend valuable goods from the attacks of offenders. ► Two complementary approaches are used: agent-based modeling and game theory. ► A Nash equilibrium is found considering the good values and the chance of success. ► The bifurcation diagram of the Nash-equilibrium as a function of the **relative** values of the goods is non-monotonic.



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# What if criminals optimize their choice? Optimal strategies to defend public goods

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## ABSTRACT

We investigate optimal strategies to defend valuable goods against the attacks of a thief. Given the value of the goods and the probability of success for the thief, we look for the strategy that assures the largest benefit to each player irrespective of the strategy of his opponent. Two complementary approaches are used: agent-based modeling and game theory. It is shown that the compromise between the value of the goods and the probability of success defines the mixed Nash equilibrium of the game, that is compared with the results of the agent-based simulations and discussed in terms of the system parameters.

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## 1. Introduction

In this note we shall study a particular example of how to protect a property from criminal actions. More precisely, we consider a classical security problem, which in its arguably simplest form can be formulated as follows: one security guard has to protect two different sites, where valuable goods are deposited, from the attack of a criminal. Each side, guard and criminal, has to select a strategy that best fits their own purpose: catching a pricey booty (and getting away with it) in one case, and keeping goods safe (and perhaps getting rid of marauders) on the other.

General qualitative results can be obtained by using a game theory approach [1–3]. Essentially, game theory seeks the best strategy a player has to play to optimize a certain payoff, for instance the largest benefit. Our problem can be defined as a zero-sum game with a payoff matrix that depends on the value of the sites and the probabilities of success for every choice of the players (guard and criminal). This is done in detail in Section 2. To shed more light on this problem, in Section 3, we address the simulation of a dynamic scenario where one guard and one thief move and fight each other for the booty. Unfortunately, even for this simple problem, agent-based models require to consider specific factors explicitly and, consequently, they need include a large number of parameters [4]. Both approaches complement each other since the information obtained from one of them allows the calibration of some of the parameters involved in the other [5]. For instance, as it will be seen, the mixed Nash equilibrium will provide information about the preferences of the agents, whereas the strategies of the agents give estimations of the probabilities of success for each of the players.

The article is organized as follows: in Section 2, we define a  $2 \times 2$  zero-sum game and study its equilibria in terms of the ratio of the value of the two sites and the probability of success of the thief for each of the four possible pairs of player choices. In Section 3 an agent-based approach is used to implement a spatial game that mimics the theoretical game. Finally, in Section 4 we compare the two approaches and conclude with some general considerations.

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## 2. The game of the two goods

Let us consider a simple mathematical game with two players: one thief  $R$  and one guard  $C$ . Since the benefits of the thief are the losses of the guard, the game is defined as a zero-sum one. The goal of the guard is to minimize his losses, whereas the goal of the thief is to maximize his benefits. Each player can choose between two sites  $A$  and  $B$ . Each choice constitutes one of their strategies.

It is assumed that the payoff each player obtains for each of the four possible pairs of strategies depends on both the probability of success of the thief if he chooses to attack site  $i$  and the guard chooses to protect site  $j$  ( $\Pi_{ij}$ ) and the value of each site ( $\alpha_i$ ). Besides, it can be supposed that the thief's success is only prevented by the encounter with the guard and thus,  $\Pi_{AB} = \Pi_{BA} = 1$ . If we rename  $\Pi_{AA} = \Pi_A$  and  $\Pi_{BB} = \Pi_B$ , the payoff matrix for the thief is given by<sup>1</sup>:

		Guard	
		$A$	$B$
Thief	$A$	$\alpha_A \Pi_A$	$\alpha_A$
	$B$	$\alpha_B$	$\alpha_B \Pi_B$

A solution of the game corresponds to a pair of strategies adopted by each player. Since each player seeks the best payoff, if one player plays at random, the other one could find a strategy that maximizes his benefits and then, in a zero-sum game, that maximizes the losses of the opponent. It is well known that for a zero-sum game at least a Nash equilibrium exists and it coincides with the one produced by maximin and minimax strategies. Indeed, minimizing the opponent's maximum payoff (minimax rule) is identical to minimizing one's own maximum loss and to maximizing one's own minimum gain (maximin rule) [6].

It seems reasonable to assume that (i) the thief  $R$  always prefers the site without surveillance (i.e. that one not chosen by the guard  $C$ ) and (ii) the guard  $C$  always prefers going to the same site chosen by the thief  $R$ . The first condition means that:

$$\alpha_B > \alpha_A \Pi_A \Rightarrow \rho = \frac{\alpha_A}{\alpha_B} < \frac{1}{\Pi_A}$$

$$\alpha_A > \alpha_B \Pi_B \Rightarrow \rho = \frac{\alpha_A}{\alpha_B} > \Pi_B.$$

Therefore, condition (i) implies that:

$$\Pi_B < \rho < \frac{1}{\Pi_A}.$$

Condition (ii) imposes the trivial inequalities:

$$\alpha_A \Pi_A < \alpha_A \Rightarrow \Pi_A < 1$$

$$\alpha_B \Pi_B < \alpha_B \Rightarrow \Pi_B < 1.$$

In order to compute the mixed Nash equilibrium, let us suppose that the thief  $R$  chooses a mixed strategy  $q_R$ , where  $q_R$  is the probability that  $R$  plays strategy  $A$ .<sup>2</sup> The payoff for the guard  $C$  is:

- $-\alpha_A \Pi_A q_R - \alpha_B (1 - q_R)$  if it chooses  $A$  and
- $-\alpha_A q_R - \alpha_B \Pi_B (1 - q_R)$  if it chooses  $B$ .

As it is well known,  $q_R$  represents a mixed Nash equilibrium if every  $C$ 's strategy is a best response to it, that is, once  $R$  has chosen  $q_R$ ,  $C$  gets the same payoff for every strategy he chooses. Therefore,

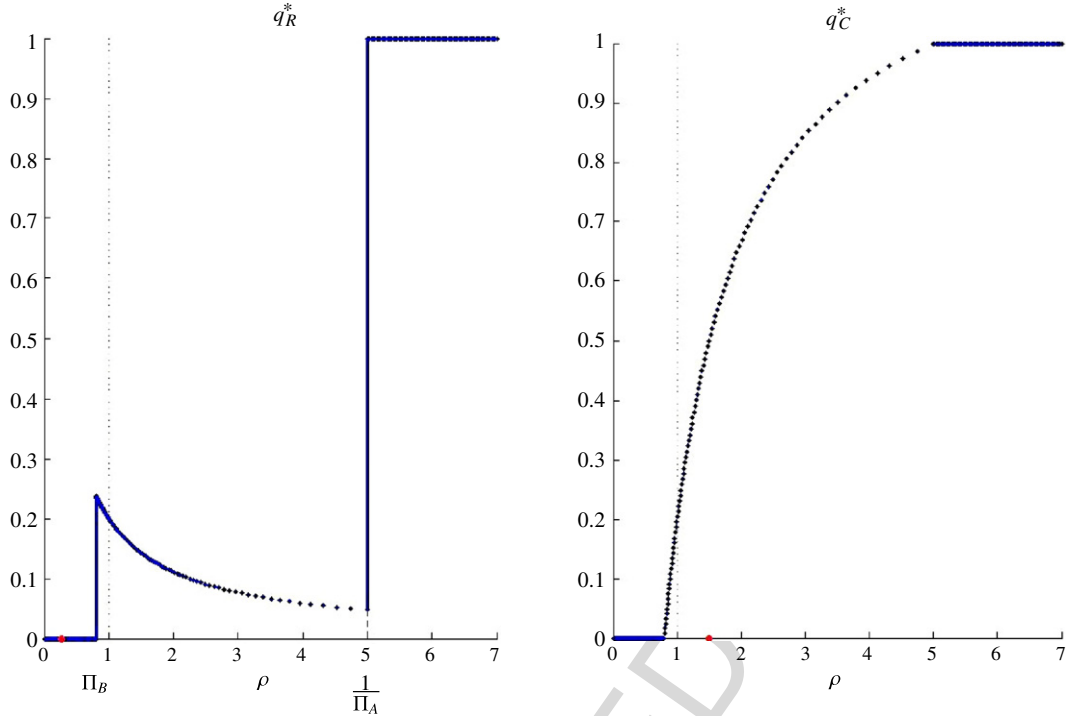
$$-\alpha_A \Pi_A q_R - \alpha_B (1 - q_R) = -\alpha_A q_R - \alpha_B \Pi_B (1 - q_R).$$

By solving this equation we obtain the mixed Nash equilibria as a function of  $\rho$  (Fig. 1):

$$q_R^* = \begin{cases} 0 & \text{if } \rho < \Pi_B \\ \left[ 0, \frac{1 - \Pi_B}{1 - \Pi_A \Pi_B} \right] & \text{if } \rho = \Pi_B \\ \frac{1 - \Pi_B}{(1 - \Pi_A) \rho + (1 - \Pi_B)} & \text{if } \Pi_B < \rho < \frac{1}{\Pi_A} \\ \left[ \frac{\Pi_A (1 - \Pi_B)}{1 - \Pi_A \Pi_B}, 1 \right] & \text{if } \rho = \frac{1}{\Pi_A} \\ 1 & \text{if } \rho > \frac{1}{\Pi_A} \end{cases}$$

<sup>1</sup> Appendix A presents a brief analysis of the general case.

<sup>2</sup>  $q_R = 1$  corresponds to pure strategy  $A$  and  $q_R = 0$  to pure strategy  $B$  (and similarly for  $C$ ).



**Fig. 1.** Graphs of  $q_R^*$  (left) and  $q_C^*$  (right) on  $\rho > 0$  with  $\alpha_A = 1$ ,  $\Pi_A = 0.2$  and  $\Pi_B = 0.8$ . The red dots on the  $\rho$ -axis are the values for which  $q_R^* = 1/2$  and  $q_C^* = 1/2$ , i.e. the values for which the player changes his preference on the places.

$$q_C^* = \begin{cases} 0 & \text{if } \rho \leq \Pi_B \\ \frac{\rho - \Pi_B}{(1 - \Pi_A)\rho + (1 - \Pi_B)} & \text{if } \Pi_B < \rho < \frac{1}{\Pi_A} \\ 1 & \text{if } \rho \geq \frac{1}{\Pi_A} \end{cases}$$

The partial derivatives of  $q_R^*$  with respect to every parameter show that it always decreases with  $\rho$  and  $\Pi_B$  and increases with  $\Pi_A$ . Moreover, it presents jumps at  $\rho = \Pi_B$  and  $\rho = 1/\Pi_A$  that indicate that the thief is indifferent to his strategies because both of them provide him the same payoff, but he will prefer the one that allows the guard to maintain this indifference. In addition,  $q_C^*$  increases with  $\rho$ , that means that the guard tends to choose A as it becomes more attractive. Therefore the thief prefers going towards B in order to avoid the guard, but if A becomes too valuable to be ignored then he goes to A even if sometimes he fails. The threshold value  $\rho = 1/\Pi_A$  represents the risk the thief can afford to take the most valuable good. Fig. 1 depicts both equilibrium probabilities  $q_R^*$  and  $q_C^*$  for the thief and the guard. Figs. 2 and 3 show a more predictable behavior: the interest of both players for a site (in particular, site A) increases if its level of protection decreases. This is obvious for the thief and thus for the guard too, who prefers to follow and hinder the thief. The corresponding payoffs of the players are given in Appendix A.

Further understanding of the behavior of the model can be obtained by considering some particular cases. When the probability of success of the thief is the same in both sites, i.e.  $\Pi_A = \Pi_B = \Pi$ , then the mixed Nash equilibria reduce for  $\Pi < \rho < 1/\Pi$  to:

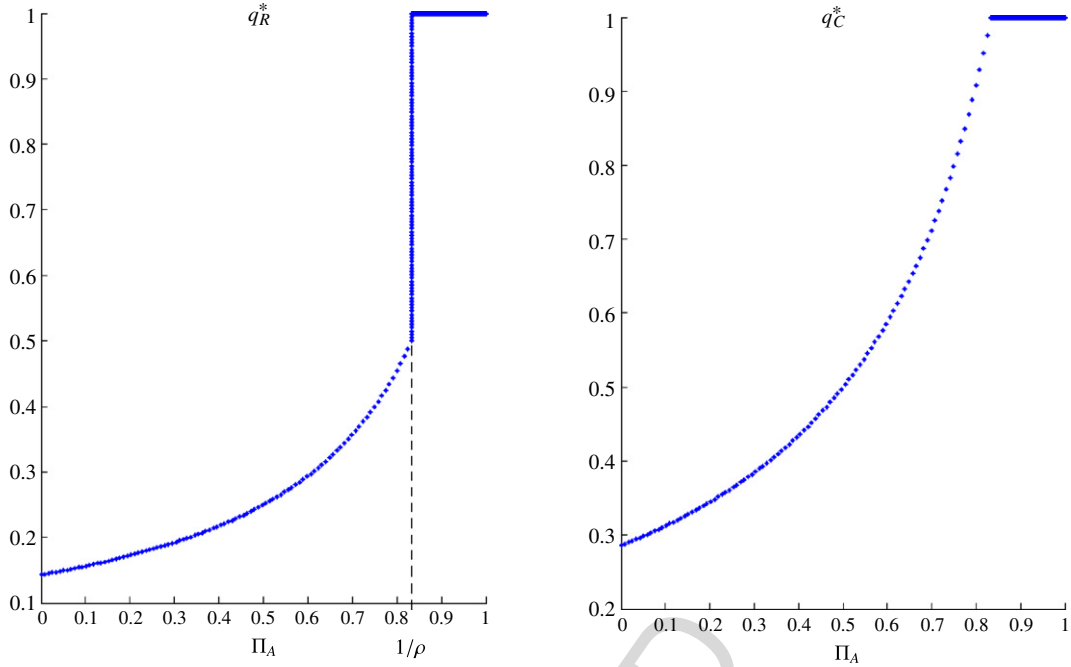
$$\begin{cases} q_R^* = \frac{1}{\rho + 1} \\ q_C^* = \frac{\rho - \Pi}{(1 - \Pi)(\rho + 1)} \end{cases}$$

As it can be seen,  $q_R^* < q_C^*$  if  $\rho > 1$ . That is, as A is more valuable than B, the preference of the guard to protect A is greater than the preference of the thief to attack this site. Besides, it can be noted that the following identity holds:

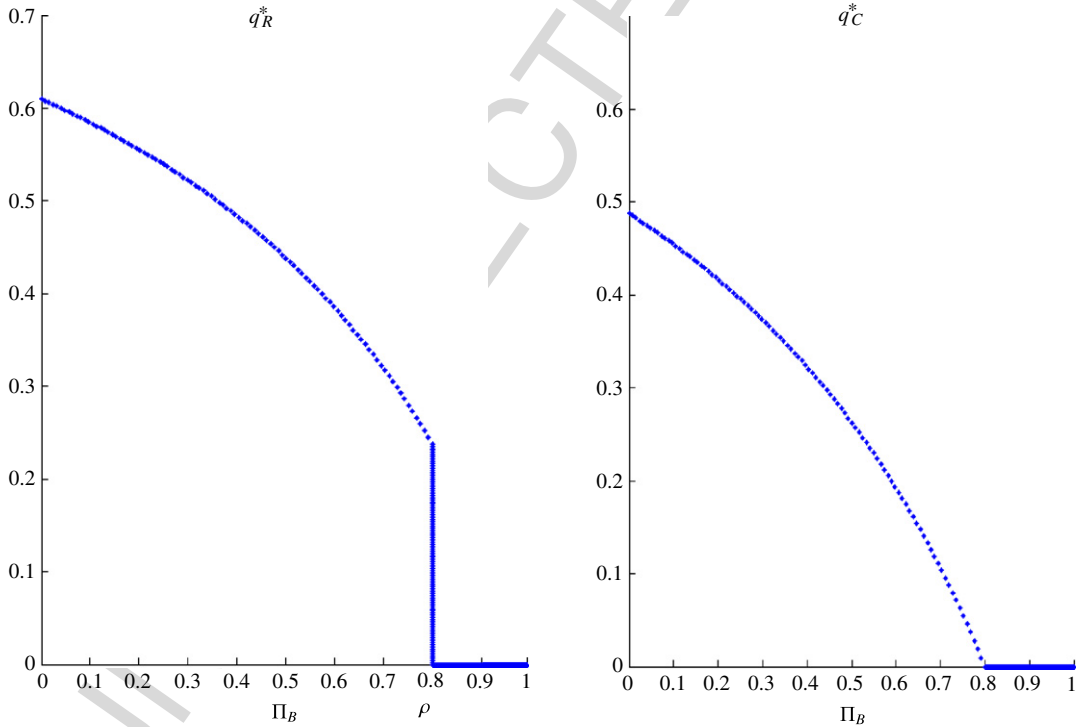
$$q_R^*|_{\rho=\Pi} + q_R^*|_{\rho=\frac{1}{\Pi}} = q_C^*|_{\rho=\Pi} + q_C^*|_{\rho=\frac{1}{\Pi}} = 1 \quad \forall \Pi$$

or equivalently, for  $q_R^*$ :

$$q_R^*|_{\rho=\Pi} = 1 - q_R^*|_{\rho=\frac{1}{\Pi}} \quad \forall \Pi.$$

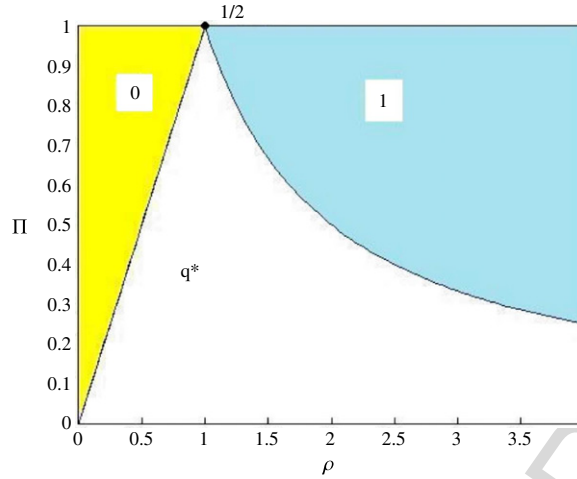


**Fig. 2.** Graphs of  $q_R^*$  (left) and  $q_C^*$  (right) on  $\Pi_A \in [0, 1]$  with  $\rho = 1.2$  and  $\Pi_B = 0.8$ .

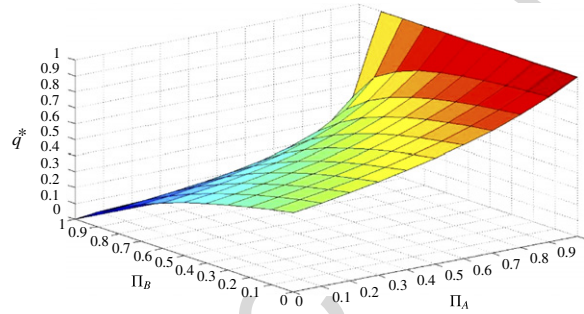


**Fig. 3.** Graphs of  $q_R^*$  (left) and  $q_C^*$  (right) on  $\Pi_B \in [0, 1]$  with  $\rho = 0.8$  and  $\Pi_A = 0.2$ .

1 Read it in this way:  $q_R^*|_{\rho=\Pi}$  is the probability of choosing A when  $\alpha_A = 1$  and  $\alpha_B = 1/\Pi$ , i.e. when A is the less attractive  
 2 site and the ratio between the most attractive goods and the less attractive one is  $1/\Pi$ ;  $1 - q_R^*|_{\rho=1/\Pi}$  is the probability of  
 3 choosing B when  $\alpha_A = 1$  and  $\alpha_B = \Pi$ , i.e. when B is the less attractive goods and the ratio between the most attractive  
 4 goods and the less attractive ones is still  $1/\Pi$ . Hence the thief has the same probability of choosing the less attractive goods  
 5 once the ratio has been fixed.



**Fig. 4.** Projection of  $q_C^*$  on  $\Pi \in [0, 1]$  and  $\rho > 0$ . In the range  $\Pi < \rho < \frac{1}{\Pi}$ ,  $q_C^* = \frac{\rho - \Pi}{(1 - \Pi)(\rho + 1)}$ . Out of this range, it takes constant values 0 and 1, respectively, for  $\rho < \Pi$  and  $\rho > 1/\Pi$ .



**Fig. 5.** Graphs of  $q_C^*$  on  $\Pi_A \in [0, 1]$  and  $\Pi_B \in [0, 1]$ .

Consider now  $q_R^*$ . It does not depend on  $\Pi$  (the thief does not choose which site to attack in terms of the probability of winning in that place because he has the same probability of success in both sites A and B) and it decreases when  $\rho$  increases, i.e. the more B decreases its value, the more the thief prefers it. Finally consider the expression of  $q_C^*$ : if  $\rho = 1$  then  $q_C^* = 1/2$  that is the case of equivalent goods randomly chosen; instead, as  $\Pi$  increases then  $q_C^*$  decreases if  $\rho < 1$  while it increases if  $\rho > 1$ . Hence, the more the thief's probability of success increases, the more the guard prefers the most attractive site. With  $\Pi$  fixed,  $q_C^*$  increases as  $\rho$  increases: hence the more B decreases its value, the more the guard prefers A. So the guard prefers to protect the most valuable goods and the thief takes advantage of this by choosing the other site, less valuable but easier to be attacked. Note that out of the range  $\Pi < \rho < \frac{1}{\Pi}$ ,  $q_C^*$  takes constant values 0 and 1, for  $\rho < \Pi$  and  $\rho > 1/\Pi$ , respectively (see Fig. 4).

Another interesting particular case occurs when  $\rho = 1$ , i.e. when the two goods have the same attractiveness  $\alpha$ . Then, for every value of  $\Pi_A$  and  $\Pi_B$ , the mixed Nash equilibrium for both players can be written as follows (see Fig. 5):

$$q^* = \frac{1 - \Pi_B}{2 - \Pi_A - \Pi_B}.$$

With  $\Pi_B$  fixed,  $q^*$  increases when  $\Pi_A$  increases, while with  $\Pi_A$  fixed,  $q^*$  decreases when  $\Pi_B$  increases. That is: both players choose the goods that the thief can get easier. Obviously, in the particular case with  $\Pi_A = \Pi_B = \Pi$  (the goods are perfectly equivalent) the following values are obtained:

$$q_R^* = q_C^* = \frac{1}{2} \quad P_R^* = -P_C^* = \frac{\alpha(1 + \Pi)}{2}.$$

Hence, both players give the same importance to the sites and gain the mean value of the payoffs for pure strategies.

Theoretical approaches face the serious handicap of calibration of the parameters. Since empirical data are scarce, computer simulations are a good alternative to obtain reference values for some parameters. The probabilities of a thief's success in sites A and B, respectively,  $\Pi_A$  and  $\Pi_B$ , are of special importance. In the next section, we present an agent-based model that simulates the spatial dynamics of two agents that allows an estimation of these two probabilities.



### 3. Simulating the game in the plane: an agent-based model

A natural implementation of the game defined in the previous section is in a bidimensional space. If we look at a city map we see that banks are located in fixed points, separated by a certain distance that could be measured in terms, for instance, of city blocks. It could be considered that two players, a thief and a guard, move on the city map and play to obtain the largest payoff. At each time step, they play a strategy according to their payoff matrices that now are a function of their position as well as the location of the (two) goods (banks). Essentially, during the simulation they are playing an attack-defend-pursuit-scape game whose dynamics can not be deduced from the theoretic game analyzed in the previous section. As we are going to see, the agent-based simulations provide a good calibration of the probabilities  $\Pi_A$  and  $\Pi_B$  for different parameter setups.

Our city map is defined as a rectangular grid formed by  $20 \times 14$  points (obtained as regular partition of each side of the rectangle) with a 1-norm distance. Formally, if  $P(x_1, x_2)$  and  $Q(x_2, y_2)$  are two points in the grid, the distance between them is given by:

$$d(P, Q) = |x_1 - x_2| + |y_1 - y_2|. \quad (1)$$

We consider two separate sites  $A$  and  $B$ <sup>3</sup> with values  $\alpha_A$  and  $\alpha_B$ , respectively, and two agents: a thief  $R$  that wants to get the goods from the sites and a guard  $C$  that protects the sites in order to minimize the possible losses. The thief (resp. guard) must decide which site he prefers to attack (resp. to defend) and, once there, he has to select the strategy to follow.

Let us denote  $q_R$  and  $q_C$  the probabilities, respectively, for the thief and for the guard to choose  $A$ . These probabilities depend on the specific characteristics of the problem. In particular, the probability  $q_R$  is a function of the values of the goods placed in each site and of the protection of the sites. If  $\rho = \alpha_A/\alpha_B$  is the ratio between the values of site  $A$  and  $B$ , we postulate that:

$$q_R = 1 - (1 - f_R)^\rho \quad (2)$$

where  $f_R$  is the preference of thief for site  $A$  when  $\alpha_A = \alpha_B$  (i.e.  $\rho = 1$ ). A similar definition applies for  $q_C$ .

Simulations can be divided in two phases. Firstly, agents just have to choose a site. Secondly, if both agents have chosen the same site, they play a spatial game. Each of them has a probability  $p_R^k$  and  $p_C^k$  to approach the chosen site  $k$  and a probability  $1 - p_R^k$  and  $1 - p_C^k$  to move randomly.<sup>4</sup> On the contrary, if the agents choose different sites, then the thief automatically wins the value of the site. A schematic description of each simulation is given in Appendix C.

Each simulation is a possible realization of the process and the outcome depends on the parameter setup used. Note that, as defined above, the goods are the agents' main interest (independently of the opponent strategy). Nevertheless, only few changes in the dynamical rules would allow to take into account also the case in which the thief simultaneously wants both to get the goods and to escape from the guard and the guard wants both to protect the goods and to catch the thief. This consideration opens the question of the role of repetition in playing this game; if the thief is not caught in his first try, he can try again but, of course, with a different risk.

In order to illustrate the behavior of the model and to compare it with the game of the two goods (Section 2), we report here the results of some simulations that reproduce the choices of the agents according to the previous scheme. Simulations were performed with a MATLAB code whose inputs are  $f_R$  and  $f_C$ ,  $p_R^A$ ,  $p_R^B$  and  $p_C^A$ ,  $p_C^B$  and  $\rho$ . Each situation (particular parameter setup) has been repeated 100 times. The game is implemented with random initial conditions to avoid their influence on the final results. Players move a unit length at each time step on the grid formed by the regular partition of the rectangle  $[-10, 10] \times [-7, 7]$ . The goods are placed at the origin of coordinates  $(0, 0)$ . The guard catches the thief if they are in the same position at the same time; the thief gets the goods if he reaches the point  $(0, 0)$  before being caught by the guard. The final time for each simulation is  $T_f = 150$ . In this period, practically no draws occur in all simulations.

We focus our study on the case where the goods placed in  $A$  are more valuable than those placed in  $B$ . In particular, we take  $\rho = \frac{3}{2}$ . As a consequence, we assume that the guard mainly prefers to defend the more valuable site. Hence, we take

$$f_C = 0.9. \quad (3)$$

Besides, we assume that the probabilities that drive the thief's movement are given by the following setup:

$$p_R^A = 0.4; \quad p_R^B = 0.8; \quad (4)$$

which represents an agent that has a different strategy depending on the site chosen: it points towards the target double the amount of times in  $B$  than in  $A$ . Similarly, we assume for the guard the following setup:

$$p_C^A = 0.8; \quad p_C^B = 0.5; \quad (5)$$

which means that the guard has a greater tendency to defend (without pursuing) in  $A$  than in  $B$ . Note that, in a certain sense, the strategies of the two players complement each other: because the guard devotes larger efforts to defend the goods in  $A$ ,

<sup>3</sup> The sites are separated in the sense that, once one agent has decided to go towards one of them, it cannot change its choice and go to the other one.

<sup>4</sup> For the guard,  $0 < p_C < 1$  represents a situation between patrolling and guarding. Similarly, a value of  $p_R < 1$  introduces some randomness in the thief movements to scape the guard.

a direct attack of the thief is quite difficult and then, he prefers to move randomly and wait for his opportunity. Instead, in site  $B$ , since the guard alternates patrolling and guardianship, the thief tries more often to **directly approach** the goods.

In what follows, we study the behavior of the system for three different values of the thief's preferences for both sites, i.e. we take  $f_R$  as the control parameter.

The thief and guard have the same preference. Assume first that thief has a strong preference for site  $A$ . Concretely, we take:

$$f_R = 0.9. \quad (6)$$

In this case, both thief and guard meet most of the **time** in site  $A$ . There, the defensive strategy of the guard allows him to catch the thief 65% of the **time**. The thief only gets the goods 35% of the **time**. Note that, as we said, no draws occur in the time interval  $[0, 150]$ .

The thief has equal preferences. Let us now assume that the thief has no special preference for any site (i.e. he does not mind about the value of goods), whereas the guard still maintains his strong preference to defend the most valuable site. Hence, we take:

$$f_R = 0.5. \quad (7)$$

In this case, the thief gets the goods 65% of the **time** and the rest of the **time** is caught by the guard. Nonetheless, since  $\alpha_A > \alpha_B$ , the total payoff of the thief is lower than in the **previous** case.

The thief has a strong preference for the less valuable site. Let us finally assume that the thief prefers to move to the site  $B$ , less protected by the guard. For instance, let us take:

$$f_R = 0.2. \quad (8)$$

This strategy could be viewed as a reasonable response to the strong preference of the guard for site  $A$ . In this case, the thief succeeds in getting the goods most of the **time**, concretely 86%, whereas he is caught only 14% of the **time**. Obviously, as we said before, this choice could be considered as successful if the final payoff of the thief is large enough. This, as we have discussed in the previous section, depends on  $\rho$ , the ratio between the value of the goods.

■ As a consequence, sometimes it seems more convenient for the thief to go toward the less precious treasure in order to avoid the encounter with his opponent, even if this means not getting the maximum benefit. Obviously, reacting to the strategies of the thief, the guard could change his preference for  $A$  or the way of protecting the goods in order to lower the number of **the** thief's successes.

To study how the system parameters affect the output of the agent-based model, we have simulated the process using different values of  $f_R$  and  $f_C$ . In Fig. 6 we can see how the number of victories of the thief changes with  $f_C$ : in Fig. 6A, when  $f_C \simeq 0$  (the guard goes almost always to  $B$ ) the thief wins more or less 80 times in 100; as  $f_C$  increases, so does the probability of success and, in the limit, for  $f_C = 1$  the thief always wins (because they choose different sites). In Fig. 6C the situation is just the contrary: when  $f_C = 0$  the thief always wins and when  $f_C \simeq 1$  he wins more or less 20 times out of 100.

Fig. 6D shows that with the strategy obtained from the theoretic game studied in Section 2 the thief wins more or less always the same number of times for every strategy chosen by the guard. We use the value of  $q_i$  obtained at the Nash equilibrium as a function of  $\rho$  and, then, we find an optimal  $f_i$  by inverting the formula (2):

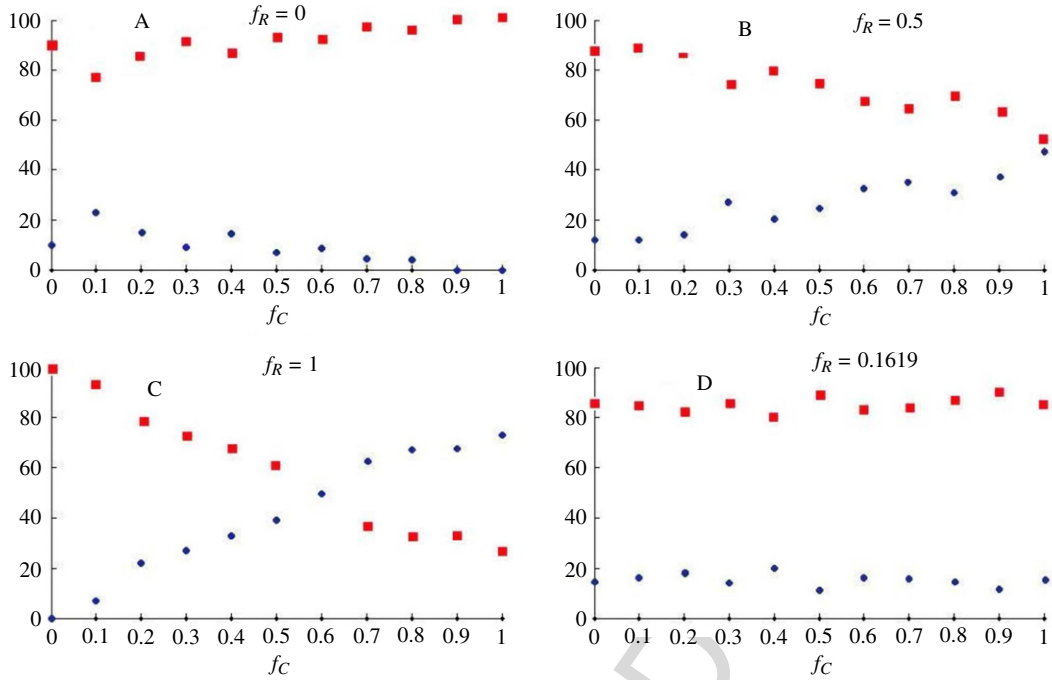
$$f_i = 1 - (1 - q_i)^{1/\rho}.$$

The agent-based model can be applied to calibrate the probabilities  $P_A$  and  $P_B$ . As before, let us assume that  $A$  is more valuable than  $B$ . Then, we can suppose that the guard has a larger preference to choose site  $A$  and to stay close to the goods than to patrol. As a consequence, we can expect that the thief has a lower preference to attack this site. Instead, when the guard is in  $B$ , he can decide also to patrol around and so the thief is more enticed to attack this site. For example, this situation can be described by the following parameter setup for the probabilities of choosing each site for the players:  $p_R^A = 0.4$  and  $p_C^A = 0.8$  and  $p_R^B = 0.8$  and  $p_C^B = 0.5$ . The second phase of the agent-based model has been implemented with these values for  $p_R$  and  $p_C$ . It turns out that the thief gets the goods 19% of the **time** in  $A$  and 78% of the **time** in  $B$ . Hence, it would yield the following values for the probabilities of victory for the thief in each site:  $\Pi_A = 0.19$  and  $\Pi_B = 0.78$ .

#### 4. Discussion and concluding remarks

The protection of valuable goods from the attack of offenders is one of the main challenges faced by security forces. This is a complex adaptive problem that depends on multiple factors, in particular on the value of the goods and the security measures that protect them. In this paper we have presented a simple model that has been studied by means of two complementary approaches: game theory and agent-based modeling.

In Section 2, we consider a game model where one guard protects two valuable sites from the attack of one thief. This is a zero-sum game whose static properties can be completely solved in terms of its mixed Nash equilibria. These equilibrium



**Fig. 6.** Output of the simulations with  $p_R^A = 0.4$ ,  $p_R^B = 0.8$ ,  $p_C^A = 0.8$ ,  $p_C^B = 0.5$ ,  $\rho = 1.1765$  and for different values of  $f_R$  varying  $f_C \in [0, 1]$ . The red squares are the times the thief gets the goods and the blue points are the times the guard catches the thief.

points are studied as a function of the ratio between the values of the goods  $\rho = \alpha_A/\alpha_B$  and the probability of success  $\Pi_{ij}$  for each of the four pairs of strategies of the game.

In Section 3 we simulate the theoretic game in a plane by means of an agent-based model. Firstly, both players have to choose between the two sites. Secondly, if both of them have chosen the same site, then they play a spatial game whose characteristics depend on the parameter setup, namely the tendency to attack (defend) or to escape (patrol) for the thief (guard), respectively. As expected, the simulations show a complex dynamics whose dependence on the system parameters is not evident.

The dependence of the Nash equilibria on  $\rho$  has a clear meaning. Assume that  $A$  and  $B$  are two metro stations and that their values are the number of passengers passing through them. Hence, it is reasonable that  $\alpha_A$  and  $\alpha_B$  vary during the day and so, to study how the equilibria react to their changes makes sense. The study of  $q_R^*$  and  $q_C^*$  in terms of the probabilities of winning for the thief is also natural. The change of  $\Pi_A$  or  $\Pi_B$  is equivalent to modifying the level of protection of the sites  $A$  and  $B$ , respectively. Therefore, knowing the Nash equilibria as a function of these two parameters allows to test the efficiency of different protection measures. The values of  $\Pi_A$  and  $\Pi_B$  that minimize the losses of the guard for a given  $\rho$  could represent the optimal protection strategies. A sound conclusion that can be derived from this study is that a thief who has to choose to steal one of two different goods will prefer the less valuable one as far as the ratio between the two values does not exceed a threshold point related to the protection level of the goods.

We would like to point out the interplay between the two techniques applied in this paper. We have seen how the use of game theory allows to obtain qualitative information about complex problems where several agents interact among them and with the environment they move in. In particular, game theory provides optimal strategies as functions of some characteristic parameters of the system. Nevertheless, in order to be calibrated, this kind of model needs real data that in the field of criminality are really hard to be obtained from any source. To overcome this difficulty, agent-based models can be used to accomplish simulations that reproduce the situation under study. In this sense, a relevant question to be addressed in future work is how the outputs of the guarding game played in the agent-based model depend on  $p_R$  and  $p_C$ . With respect to the guard, varying this parameter means choosing different protection strategies: from patrolling ( $p_C = 0$ ) to fixed guarding ( $p_R = 1$ ). To know which mixed strategy would produce a minimum loss is of great interest for police when fighting crime. Finally, the model could be generalized considering a team of  $m$  guards that has to protect  $n$  sites: the game becomes a multi-player one in which guards and thieves form two teams (coordinated or not) and we could investigate how cooperation within each team can improve its own payoff.

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The Authors are indebted to Miguel A. Herrero for many interesting discussions and helpful comments. This work has been performed in the framework of the activities of the *Instituto de Matemática Interdisciplinar* and partially supported by

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## Appendix A

The payoff matrix for the thief is then defined as follows:

		Guard	
		A	B
Thief	A	$\alpha_A \Pi_{AA}$	$\alpha_A \Pi_{AB}$
	B	$\alpha_B \Pi_{BA}$	$\alpha_B \Pi_{BB}$

It is not difficult to obtain the value of  $q_R$  at the mixed Nash equilibrium:

$$q_R^* = \frac{\Pi_{BA} - \Pi_{BB}}{(\Pi_{AB} - \Pi_{AA})\rho + (\Pi_{BA} - \Pi_{BB})}.$$

The value of the probability that the guard C plays A,  $q_C$  can be computed in a similar way:

$$q_C^* = \frac{\rho \Pi_{AB} - \Pi_{BB}}{(\Pi_{AB} - \Pi_{AA})\rho + (\Pi_{BA} - \Pi_{BB})}.$$

Note that  $q_R^* \in [0, 1] \forall \rho$ , while  $q_C^* \in [0, 1]$  for  $\rho \in [\frac{\Pi_{BB}}{\Pi_{AB}}, \frac{\Pi_{BA}}{\Pi_{AA}}]$ . Out of this range, we obtain the dominant strategies  $q_C^* = 0$  when it is negative and  $q_C^* = 1$  when it is larger than 1. If  $\rho > \Pi_{BA}/\Pi_{AA}$ , then  $\alpha_B \Pi_{BA} < \alpha_A \Pi_{AA}$  and A becomes a dominant strategy for R, i.e.  $q_R^* = 1$ . If  $\rho < \Pi_{BB}/\Pi_{AB}$ , then  $\alpha_A \Pi_{AB} < \alpha_B \Pi_{BB}$  and B becomes a dominant strategy for R, i.e.  $q_R^* = 0$ .

In particular, when  $\rho = \Pi_{BB}/\Pi_{AB}$  the payoffs of the thief R for the guard playing B are equal for both pure strategies of R. So, the thief is indifferent to which strategy to follow, but he will prefer A with a 'little' probability in order to induce C to play B. A 'little' probability means:

$$q_R^* \leq \frac{\Pi_{BA} - \Pi_{BB}}{(\Pi_{AB} - \Pi_{AA})\rho + (\Pi_{BA} - \Pi_{BB})} \Big|_{\rho = \frac{\Pi_{BB}}{\Pi_{AB}}}.$$
 (9)

Therefore,

$$q_R^* \left( \rho = \frac{\Pi_{BB}}{\Pi_{AB}} \right) = \left[ 0, \frac{\Pi_{AB}(\Pi_{BA} - \Pi_{BB})}{\Pi_{AB}\Pi_{BA} - \Pi_{AA}\Pi_{BB}} \right].$$
 (10)

Similarly,

$$q_R^* \left( \rho = \frac{\Pi_{BA}}{\Pi_{AA}} \right) = \left[ \frac{\Pi_{AA}(\Pi_{BA} - \Pi_{BB})}{\Pi_{AB}\Pi_{BA} - \Pi_{AA}\Pi_{BB}}, 1 \right].$$
 (11)

Here,  $[, ]$  means a multivalued function.

Using these expressions of  $q_R^*$  and  $q_C^*$ , the payoffs of the players in equilibrium when  $\frac{\Pi_{BB}}{\Pi_{AB}} < \rho < \frac{\Pi_{BA}}{\Pi_{AA}}$  are given by:

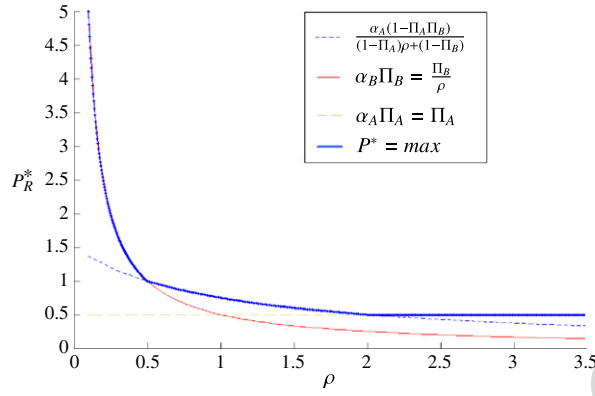
$$\begin{cases} P_R^* = \frac{\alpha_A(\Pi_{AB}\Pi_{BA} - \Pi_{AA}\Pi_{BB})}{(\Pi_{AB} - \Pi_{AA})\rho + (\Pi_{BA} - \Pi_{BB})} \\ P_C^* = -\frac{\alpha_A(\Pi_{AB}\Pi_{BA} - \Pi_{AA}\Pi_{BB})}{(\Pi_{AB} - \Pi_{AA})\rho + (\Pi_{BA} - \Pi_{BB})} \end{cases}.$$

When  $\rho < \Pi_{BB}/\Pi_{AB}$ , then  $q_R^* = q_C^* = 0$  and  $P_R^* = -P_C^* = \alpha_B \Pi_{BB}$ , whereas if  $\rho > \Pi_{BA}/\Pi_{AA}$  then  $q_R^* = q_C^* = 1$  and  $P_R^* = -P_C^* = \alpha_A \Pi_{AA}$ . Note that this function is continuous but not derivable in  $\rho = \Pi_{BB}/\Pi_{AB}$  and  $\rho = \Pi_{BA}/\Pi_{AA}$ .

In general, it is reasonable to have  $\max\{\Pi_{AA}, \Pi_{BB}\} < \min\{\Pi_{AB}, \Pi_{BA}\}$  because the presence of the guard in the sites reduces the probability of success for the thief. When, in particular, it is supposed that  $\Pi_{AB} = \Pi_{BA} = 1$  we obtain the results presented in Section 2. For this case, it is straightforward to compute the payoffs of the players in the Nash equilibrium:

$$P_R^* = -P_C^* = \begin{cases} \alpha_B \Pi_B & \text{if } \rho \leq \Pi_B \\ \frac{\alpha_A(1 - \Pi_A \Pi_B)}{(1 - \Pi_A)\rho + (1 - \Pi_B)} & \text{if } \Pi_B < \rho < \frac{1}{\Pi_A} \\ \alpha_A \Pi_A & \text{if } \rho \geq \frac{1}{\Pi_A} \end{cases}.$$

This function is continuous with respect to every parameter, but it is not derivable for  $\rho = \Pi_B$  and  $\rho = 1/\Pi_A$ . Moreover, in agreement with the theory of zero-sum games,  $P_R^*$  is the function that maximizes the income of the thief and the losses of the guard (see Fig. 7).



**Fig. 7.** Graphs of  $P_R^* = -P_C^*$  on  $\rho > 0$  with  $\Pi_A = \Pi_B = 0.5$ . It is clearly shown that the income of the thief and the losses of the guard are maximized.

## Appendix B

The game presented is perfectly equivalent to the one in which C starts having both goods ( $\alpha_A + \alpha_B$ ) and at the end gains what R did not steal:

		Guard	
		A	B
Thief	A	$\alpha_A \Pi_A, \alpha_A(1 - \Pi_A) + \alpha_B$	$\alpha_A, \alpha_B$
	B	$\alpha_B, \alpha_A$	$\alpha_B \Pi_B, \alpha_A + \alpha_B(1 - \Pi_B)$

The previous zero-sum game has been transformed more generally into a constant-sum one, where the benefits and losses of all players sum up to the same value  $\alpha_A + \alpha_B$  for every outcome. Since payoffs can always be normalized, constant-sum games may be represented as (and are equivalent to) zero-sum games.

The two games described are equivalent in the sense that the same expressions for the mixed Nash equilibrium, computed in the previous section with the first payoff matrix, are found using this new payoff matrix.

What is changing is the payoff of the guard: if  $P_C^{(1)}$  is the payoff found with the first payoff matrix and  $P_C^{(2)}$  is that one found with the new matrix, then  $P_C^{(1)} + \alpha_A + \alpha_B = P_C^{(2)}$ .

If players use mixed strategies ( $\Pi_B < \rho < \frac{1}{\Pi_A}$ ), it can be found:

$$P_C^{(2)} = \frac{\alpha_A[(1 - \Pi_A)\rho + (1 - \Pi_B)(1 - \Pi_A + 1/\rho)]}{(1 - \Pi_A)\rho + 1 - \Pi_B}.$$

## Appendix C

The strategies of the players during this spatial game are implemented as follows:

- Guard
  1. if  $\text{rand}(1)^5 \leq q_C$   
choose site A
  - else  
choose site B
  2. if  $\text{rand}(1) \leq p_C$   
minimize the distance from the goods chosen
  - else  
patrol around the goods chosen moving randomly.
- Thief
  1. if  $\text{rand}(1) \leq q_R$   
choose site A
  - else  
choose site B

<sup>5</sup>  $\text{textitrand}(1)$  is a random number chosen from a uniform distribution on the interval (0, 1).

2. if  $\text{rand}(1) \leq p_R$   
     minimize the distance to the goods chosen  
   else  
     move randomly around the goods chosen.

Here, “minimize the distance from/to the goods chosen” means that the agent moves to the point that is closest to the goods (compared with its neighbor points). When the agent “moves randomly around the goods”, he selects one of the four points of his nearest neighborhood by chance and moves towards it.

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